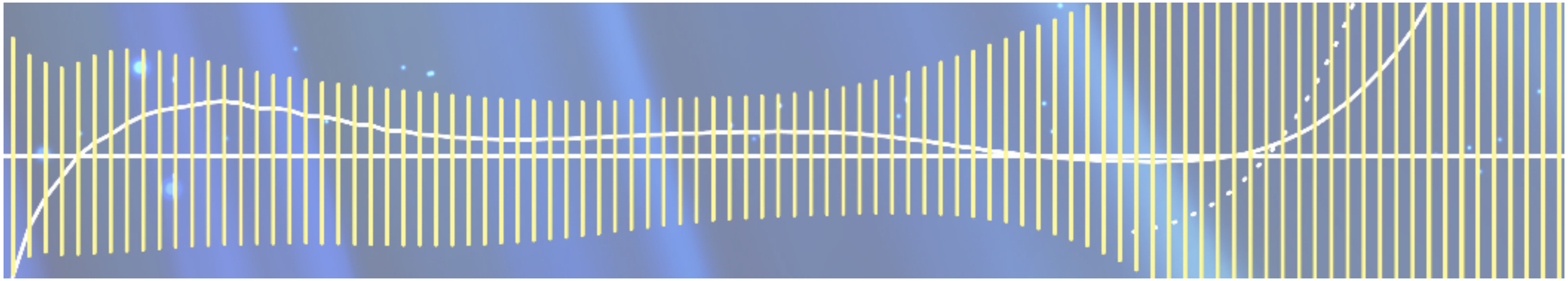

Higher moments of PDFs in lattice QCD

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Jefferson Lab

3rd International Workshop on Nucleon Structure at Large Bjorken x

- Lattice QCD and hadron structure
- The problem with higher moments (~ high x)
- Possible avenues for progress
- ~~New results~~

Hadron structure in LQCD

- Lattice techniques are the only (known) way to access non-perturbative hadron structure from QCD
- LQCD formulated in Euclidean space to use importance sampling for functional integrals: SO(3,1)→O(4)
  - Light cone physics is not apparent
  - Example: unpolarised pdf

$$q(x,\mu) = \langle p|\mathcal{O}^q_\gamma(x)|p\rangle$$
$$= \int_{-\infty}^{\infty}\frac{dz}{2\pi}\, e^{izx}\left\langle p\left|\overline{q}\left(-\frac{z}{2}n\right) n\cdot\gamma\, \mathcal{U}_{[-\frac{z}{2}n,\frac{z}{2}n]} q\left(\frac{z}{2}n\right)\right|p\right\rangle$$

- Wilson operator product expansion to the rescue

$$\mathcal{U}_{[-\frac{z}{2}n,\frac{z}{2}n]} = \mathcal{P}\text{exp}\left[ig\int_{z/2}^{-z/2} n\cdot A(z'n)dz'\right]$$

Hadron structure in LQCD

- OPE for the case in question

$$q(x,\mu) = \langle p|\mathcal{O}^q_\gamma(x)|p\rangle$$
$$= \int_{-\infty}^{\infty} \frac{dz}{2\pi}\, e^{izx} \left\langle p \left| \overline{q}\left(-\frac{z}{2}n\right) n\cdot\gamma\, \mathcal{U}_{[-\frac{z}{2}n,\frac{z}{2}n]} q\left(\frac{z}{2}n\right)\right| p\right\rangle$$

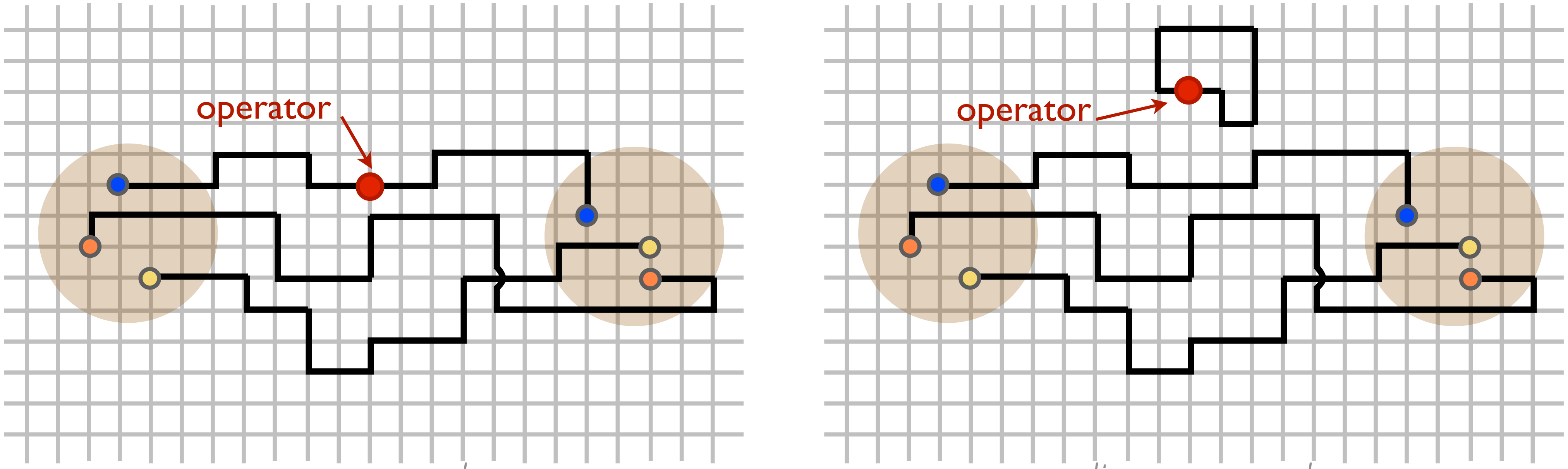
$$\langle p|\overline{q}\gamma^{\{\mu_0}D^{\mu_1}\dots D^{\mu_n\}}q|p\rangle = \langle x^n\rangle_q\,[p^{\mu_0}\dots p^{\mu_n} - \mathrm{traces}]$$

$$\langle x^n\rangle_q = \int_{-1}^{1} dx\, x^n q(x)$$

- Non-perturbative matrix elements can be calculated
  - Determine Mellin moments of PDFs

Lattice technology

- Measuring hadronic matrix elements has a long history [Martinelli & Sachrajda 86]
- Measure two- and three-point correlations on an ensemble of gauge configurations



connected *disconnected*

- Ratio C_3/C_2 proportional to lattice matrix element

Lattice technology

- Connect to experimental measurements: bare matrix element → continuum renormalisation scheme

$$\mathcal{O}_i^{cont} = Z_i \mathcal{O}_i^{lat} + \sum_j Z_{ij} \mathcal{O}_j^{lat}$$

Sum over all operators with right quantum numbers

  - Simple cases: just a multiplicative scaling
  - Operator mixing also possible: e.g. flavour singlet operators mix with gluon operators

- Renormalisation constants can be calculated perturbatively (☹) or non-perturbatively (☺)

Overview of lattice results

[See Dru Renner's talk yesterday]

- QCDSF, LHP, ETMC and RBCK, ... collaborations
- Lowest three moments of $q(x)$, $\Delta q(x)$, $\delta q(x)$
- Also calculate form factors, moments of GPDs, and meson and baryon distribution amplitudes
- Continuum, chiral and infinite volume extrapolations getting sophisticated
- Only isovector moments calculated rigourously : flavour singlet moments require disconnected contractions
- Still some discrepancies

The problem at high x_{Bj}

- LQCD necessarily formulated on a discrete geometry, typically 4d hypercube: O(4)→H(4)
- Operators classified by H(4) quantum numbers
  - Finite number of irreducible representations
- Operator mixing:

$$\mathcal{O}_i^{cont} = Z_i \mathcal{O}_i^{lat} + \sum_j Z_{ij} \mathcal{O}_j^{lat}$$

Sum over all operators with right quantum numbers

- Allows (forces) mixing with operators of lower dimension
  - Coefficients scale with inverse powers of lattice cutoff Taking the continuum limit is difficult

Hypercubic group

- H(4) (aka W_4, D_4,...): finite group of rotations by $\boldsymbol{\pi}/2$ and reflections

$$H(4) = \{(a, \pi) | a \in \mathbb{Z}_2^4,\ \pi \in S_4\}$$

- 20 irreducible representations

$$4 \cdot \mathbf{1} \oplus 2 \cdot \mathbf{2} \oplus 4 \cdot \mathbf{3} \oplus 4 \cdot \mathbf{4} \oplus 4 \cdot \mathbf{6} \oplus 2 \cdot \mathbf{8}$$

Nice Example

- Continuum operator $\mathcal{O}_{\mu\nu} = \overline{q}\gamma_{\{\mu}D_{\nu\}}q$ belongs to

$$\left(\tfrac{1}{2},\tfrac{1}{2}\right) \otimes \left(\tfrac{1}{2},\tfrac{1}{2}\right) = (0,0) \oplus [(1,0) \oplus (0,1)] \oplus (1,1)$$

- Hypercubic decomposition

$$\mathbf{4}_1 \otimes \mathbf{4}_1 = \mathbf{1}_1 \oplus \mathbf{3}_1 \oplus \mathbf{6}_1 \oplus \mathbf{6}_3$$

- Lattice operators (symmetric traceless):

$$\mathcal{O}_{14} + \mathcal{O}_{41}, \qquad \mathcal{O}_{44} - \frac{1}{3}\left(\mathcal{O}_{11} + \mathcal{O}_{22} + \mathcal{O}_{33}\right)$$

- Have same continuum limit (6_3 requires **p**$\neq$0)
- No operators of lower dimension (☺)

Nasty example

- Continuum operator $\mathcal{O}_{\{\mu\nu\rho\}} = \overline{q}\gamma_{\{\mu}D_\nu D_{\rho\}}q$ lives in

$$\left(\tfrac{1}{2},\tfrac{1}{2}\right)\otimes\left(\tfrac{1}{2},\tfrac{1}{2}\right)\otimes\left(\tfrac{1}{2},\tfrac{1}{2}\right) = 4\cdot\left(\tfrac{1}{2},\tfrac{1}{2}\right)\oplus 2\cdot\left(\tfrac{3}{2},\tfrac{1}{2}\right)\oplus 2\cdot\left(\tfrac{1}{2},\tfrac{3}{2}\right)\oplus\left(\tfrac{3}{2},\tfrac{3}{2}\right)$$

- Hypercubic decomposition

$$\mathbf{4}_1\otimes\mathbf{4}_1\otimes\mathbf{4}_1 = 4\cdot\mathbf{4}_1\oplus\mathbf{4}_2\oplus\mathbf{4}_4\oplus 3\cdot\mathbf{8}_1\oplus 2\cdot\mathbf{8}_2$$

- Lattice operators:

$$\mathcal{O}_{111},\qquad \mathcal{O}_{\{123\}},\qquad \mathcal{O}_{\{441\}}-\tfrac{1}{2}\left(\mathcal{O}_{\{221\}}+\mathcal{O}_{\{331\}}\right)$$

- Same continuum limit but $\mathcal{O}_{111}$ mixes with $\overline{q}\gamma_1 q \in \mathbf{4_1}$ and the coefficient absorbs the missing dimensions (☹)
  - Always the case for all n > 4 operators

Moments ⇒ PDFs

- Well defined problem: inverse Mellin transform
  - Requires all integer moments
- How useful are just 3 moments?
  - Fit parametric form for PDF
  - Standard parameterisations have ~5 parameters for a given PDF (☹)
- A different approach is needed

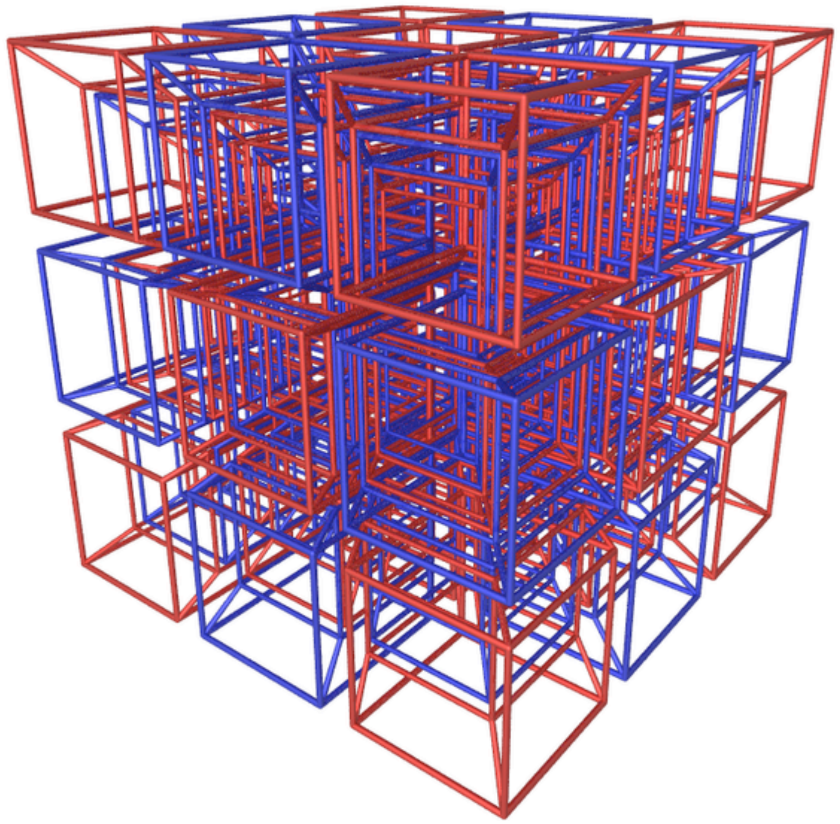
NB: 3 moments is 2 of q_v and 1 of $q+\bar{q}$

Options

- Possible methods to bypass this problem
  - Alternate discretisations
  - Brute force
  - OPE without OPE
  - Hamiltonian/transverse lattice approaches (PDFs directly) [Dalley & Burkardt; Grünewald, Ilgenfritz & Pirner; Vary et al.]
  - ??

Alternate discretisations

- Large-scale LQCD universally formulated on a hypercubic geometry
- 4D "fcc" geometry (F_4) has larger symmetry
  - Some higher moments would be accessible
  - Entirely new simulations (and algorithms) needed
  - Problems with fermions



Brute force approach

- The generic lattice mixing problem

$$\mathcal{O}_i^{cont} = Z_i \mathcal{O}_i^{lat} + \sum_j Z_{ij} \mathcal{O}_j^{lat}$$

where Zij ~ a^{-m} (m>0) may not be insurmountable

- Precise calculations of large set of operators at a range of *a* could work
- Examples: kaon weak matrix elements, d_2 [QCDSF]
- Gets worse as n increases and hopeless for GPDs

OPE in Euclidean space

[WD, CJD Lin, Phys.Rev. D73 (2006) 014501]

- OPE of Compton tensor can be studied directly in Euclidean space
  1. Calculate current-current commutator
  2. Extrapolate to continuum - restore O(4)
  3. Match to Euclidean OPE to extract matrix elements of local operators
- Determines the same moments as in Minkowski space.
- Analytic continuation is in Wilson coefficients
- Still a very difficult calculation [Schierholz '99]

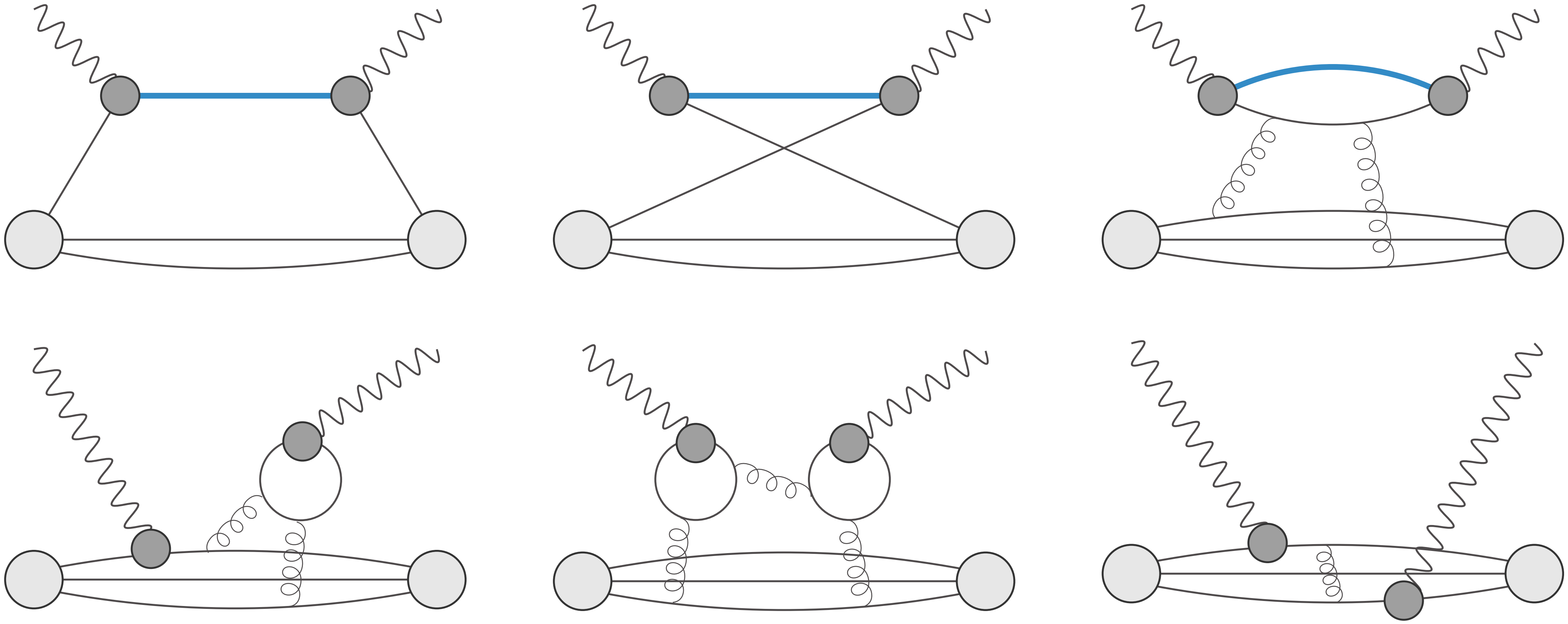
[see also Braun & Mueller Eur.Phys.J. C55 (2008) 349]

Fictitious heavy quarks

- Consider Compton tensor for currents that couple light quarks to heavy fictitious quark Ψ
- No disconnected contractions ➡ practical
- Heavy quark mass acts like photon virtuality to suppress higher twists
- Heavy quark integrated out after OPE
    - Determine same PDF moments
    - Effects in perturbative Wilson coefficients

Lattice correlators

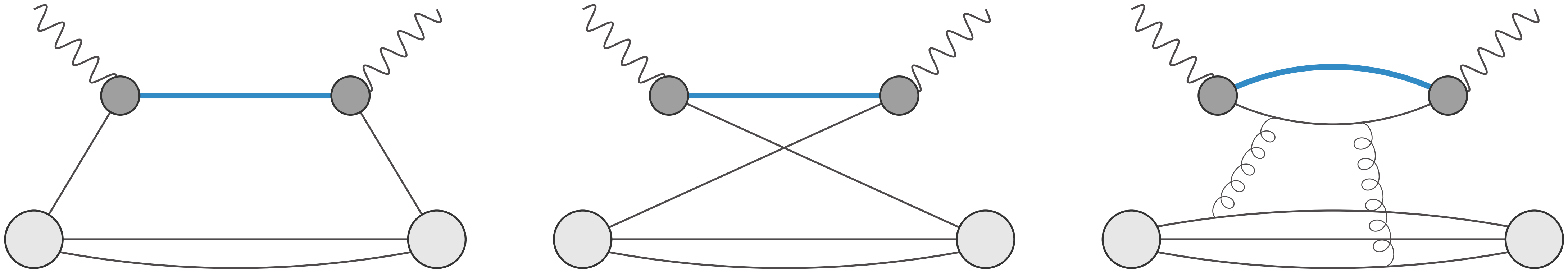
- Quark contractions in Compton correlator



- Greatly simplified with heavy quark: no disconnected contributions in isovector case

Lattice correlators

- Quark contractions in Compton correlator



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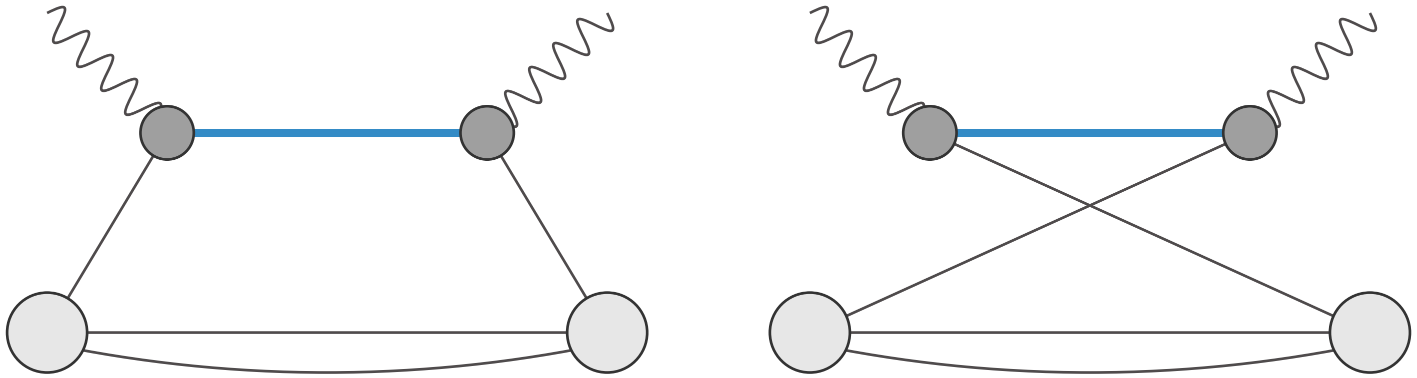
Lattice correlators

- Quark contractions in Compton correlator



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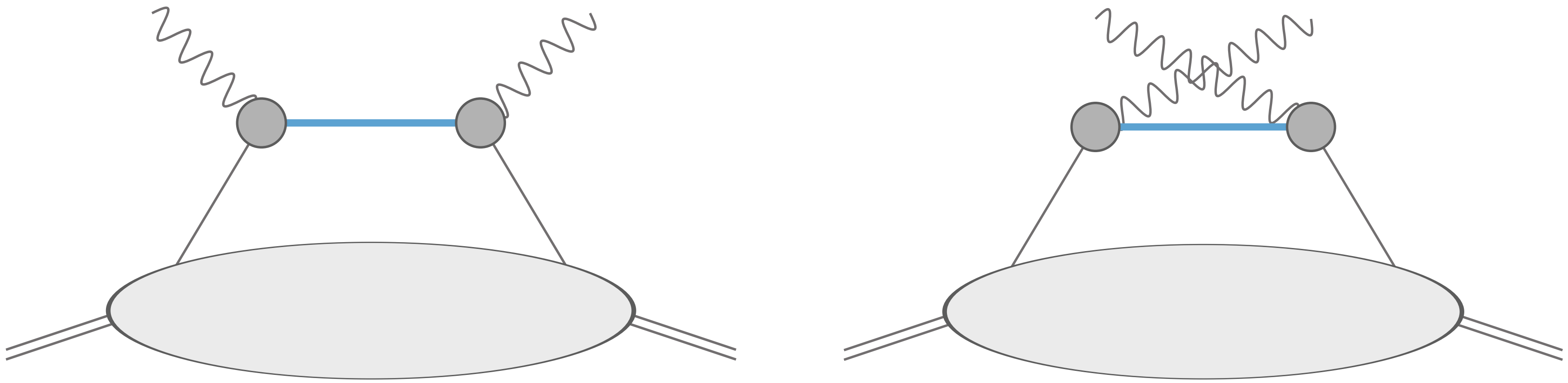
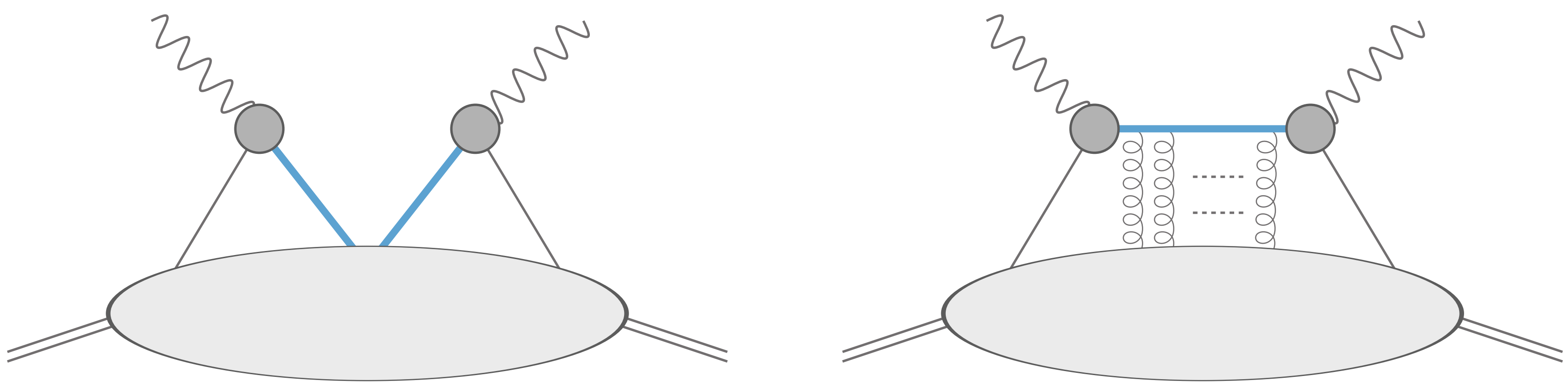
Heavy quark Compton tensor

- Leading twist contributions



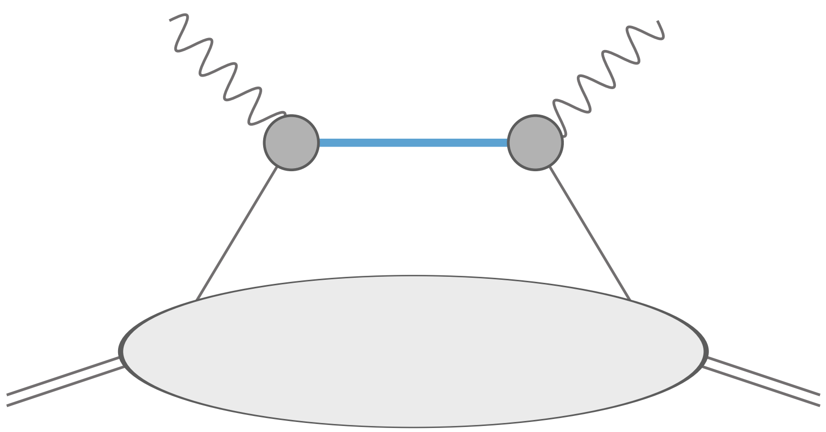
- Higher twist contributions

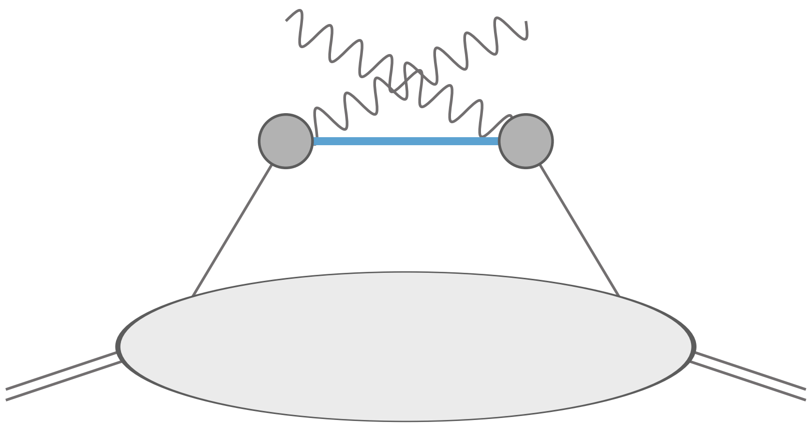


- Significantly reduced by heavy quark

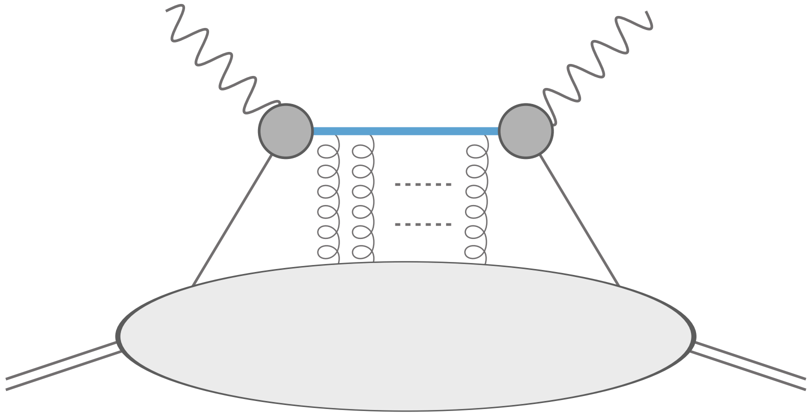
Heavy quark Compton tensor

- Leading twist contributions





- Higher twist contributions



- Significantly reduced by heavy quark

Heavy quark DIS

- Compton tensor with heavy quark

$$T^{\mu\nu}(p,q) = \sum_S \int d^4x\ \mathrm{e}^{iq\cdot x} \langle p, S | T\left[J^\mu_{\Psi,\psi}(x) J^\nu_{\Psi,\psi}(0) \right] | p, S \rangle$$

- Heavy-light vector current $J^\mu_{\Psi,\psi} = \overline{\psi}\Gamma^\mu\Psi + \overline{\Psi}\Gamma^\mu\psi$
- Operator product expansion: heavy propagator

$$\overline{\psi}\frac{-i\left(i\overleftrightarrow{\not{D}} + \not{q}\right) + m_\Psi}{(i\overleftrightarrow{D} + q)^2 + m_\Psi^2}\psi = -\overline{\psi}\frac{-i\left(i\overleftrightarrow{\not{D}} + \not{q}\right) + m_\Psi}{Q^2 + \overleftrightarrow{D}^2 - m_\Psi^2}\sum_{n=0}^{\infty}\left(\frac{-2i\ q\cdot\overleftrightarrow{D}}{Q^2 + \overleftrightarrow{D}^2 - m_\Psi^2}\right)^n\psi$$

- Denominator ⇒ $\tilde{Q}^2 = Q^2 - M_\Psi^2 + \alpha M_\Psi + \beta$
- <u>Higher twists?</u>

Heavy-light meson mass

Binding energy ~Λ

Higher twists ~Λ^2

Heavy quark DIS

- Compton tensor with heavy quark

$$T^{\mu\nu}(p,q) = \sum_S \int d^4x\ \mathrm{e}^{iq\cdot x} \langle p,S| T\left[J^{\mu}_{\Psi,\psi}(x) J^{\nu}_{\Psi,\psi}(0)\right] |p,S\rangle$$

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- Denominator ⇒ $\tilde{Q}^2 = Q^2 - M_\Psi^2 + \alpha M_\Psi + \beta_{\mathbf{n}}$
  - Heavy-light meson mass (M_Ψ^2)
  - Binding energy ~Λ (αM_Ψ)
  - Higher twists ~Λ^2 ($\beta_{\mathbf{n}}$)
- <u>Higher twists?</u>

Heavy quark DIS

- Usual twist-two operators

$$\mathcal{O}_{\psi}^{\mu_1\ldots\mu_n} = \overline{\psi}\gamma^{\{\mu_1}\left(i\overleftrightarrow{D}^{\mu_2}\right)\ldots\left(i\overleftrightarrow{D}^{\mu_n\}}\right)\psi - \text{traces}$$

- Twist-3 scalar operators also contribute: e(x)

$$\widehat{\mathcal{O}}_{\psi}^{\mu_1\ldots\mu_n} = \overline{\psi}\left(i\overleftrightarrow{D}^{\mu_1}\right)\ldots\left(i\overleftrightarrow{D}^{\mu_n}\right)\psi - \text{traces}$$

- Matrix elements

$$\sum_S \langle p, S|\mathcal{O}_{\psi}^{\mu_1\ldots\mu_n}|p, S\rangle = A_{\psi}^{n}(\mu^2)\left[p^{\mu_1}\ldots p^{\mu_n} - \text{traces}\right]$$

$$\sum_S \langle p, S|\widehat{\mathcal{O}}_{\psi}^{\mu_1\ldots\mu_n}|p, S\rangle = i\ M\ \widehat{A}_{\psi}^{n}(\mu^2)\left[p^{\mu_1}\ldots p^{\mu_n} - \text{traces}\right]$$

Heavy quark DIS

- Leads to

Wilson coefficients

Gegenbauer polynomials

PDF moments

$$T^{\{\mu\nu\}}_{\Psi,\psi} = i \sum_{n=2,\text{even}}^{\infty} A^n_\psi(\mu)\zeta^n \Bigg\{ \delta^{\mu\nu} \left[\mathcal{C}_n \frac{\widetilde{Q}^2}{q^2} \frac{n C^{(1)}_n(\eta) - 2\eta C^{(2)}_{n-1}(\eta)}{n(n-1)} + \mathcal{C}'_n C^{(1)}_n(\eta) \right] + \frac{p^\mu p^\nu \widetilde{Q}^2}{(p\cdot q)^2} \mathcal{C}_n \left[\frac{8\eta^2 C^{(3)}_{n-2}(\eta)}{n(n-1)} \right]$$

$$+4\frac{p^{\{\mu}q^{\nu\}}}{p\cdot q} \left[\mathcal{C}_n \frac{\widetilde{Q}^2}{q^2} \frac{(n-1)\eta C^{(2)}_{n-1}(\eta) - 4\eta^2 C^{(3)}_{n-2}(\eta)}{n(n-1)} - \mathcal{C}''_n \frac{\eta}{n} C^{(2)}_{n-1}(\eta) \right]$$

$$+\frac{q^\mu q^\nu}{q^2} \left[\mathcal{C}_n \frac{\widetilde{Q}^2}{q^2} \frac{n(n-2)C^{(1)}_n(\eta) - 2\eta(2n-3)C^{(2)}_{n-1}(\eta) + 8\eta^2 C^{(3)}_{n-2}(\eta)}{n(n-1)} - 2\mathcal{C}''_n \left(C^{(1)}_n(\eta) - 2\frac{\eta}{n} C^{(2)}_{n-1}(\eta) \right) \right] \Bigg\}$$

$$-2i\frac{M(m_\Psi - m)}{\widetilde{Q}^2} \delta^{\mu\nu} \sum_{n=0,\text{even}}^{\infty} \widehat{\mathcal{C}}_n \widehat{A}^n_\psi(\mu)\zeta^n C^{(1)}_n(\eta)$$

- where $\zeta = \frac{\sqrt{p^2q^2}}{\widetilde{Q}^2}$ $\eta = \frac{p\cdot q}{\sqrt{p^2q^2}}$
- Gegenbauer polynomials from target mass corrections: powers of $\left[p^2/q^2\right]^j$

Heavy quark DIS

- ... in a more comprehensible form (target rest frame)

$$T^{\{34\}}_{\Psi,\psi}(p,q) = \sum_{n=2,\mathrm{even}}^{\infty} A^n_\psi(\mu) f(n)$$

- where *f(n)* is completely known:

$$f(n) = -\sqrt{q_0^2 - Q^2}\,\zeta^n \left\{ \frac{2}{q_0} \left[\mathcal{C}_n \frac{\widetilde{Q}^2}{Q^2} \frac{(n-1)\eta C^{(2)}_{n-1}(\eta) - 4\eta^2 C^{(3)}_{n-2}(\eta)}{n(n-1)} + \ldots \right.\right.$$

  - Kinematic variables & Wilson coefficients
  - Parameters $\alpha,\ \beta$

known perturbatively

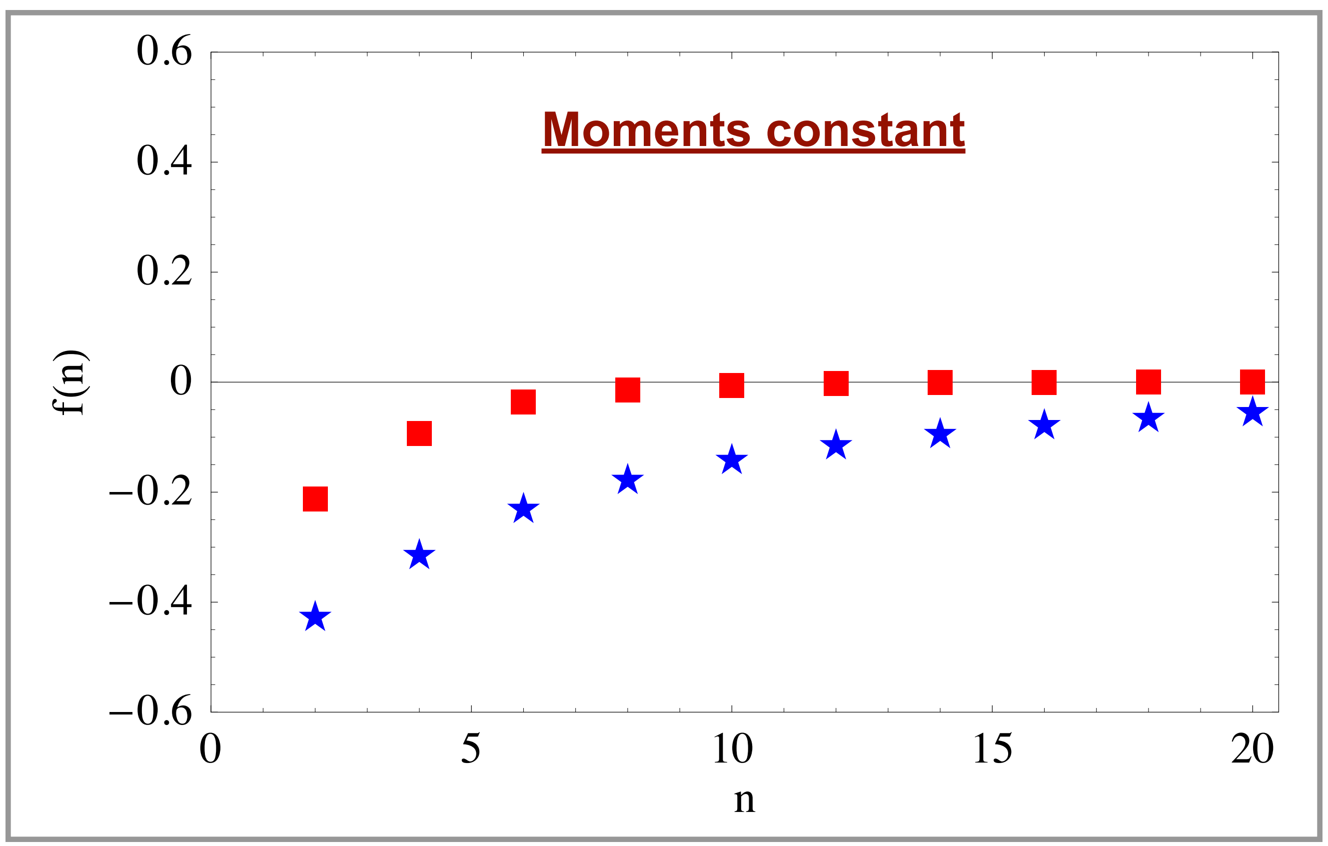
- Fits to calculations of LHS ⇒ $A^n_\psi(\mu),\ \alpha,\ \beta$

Lattice details

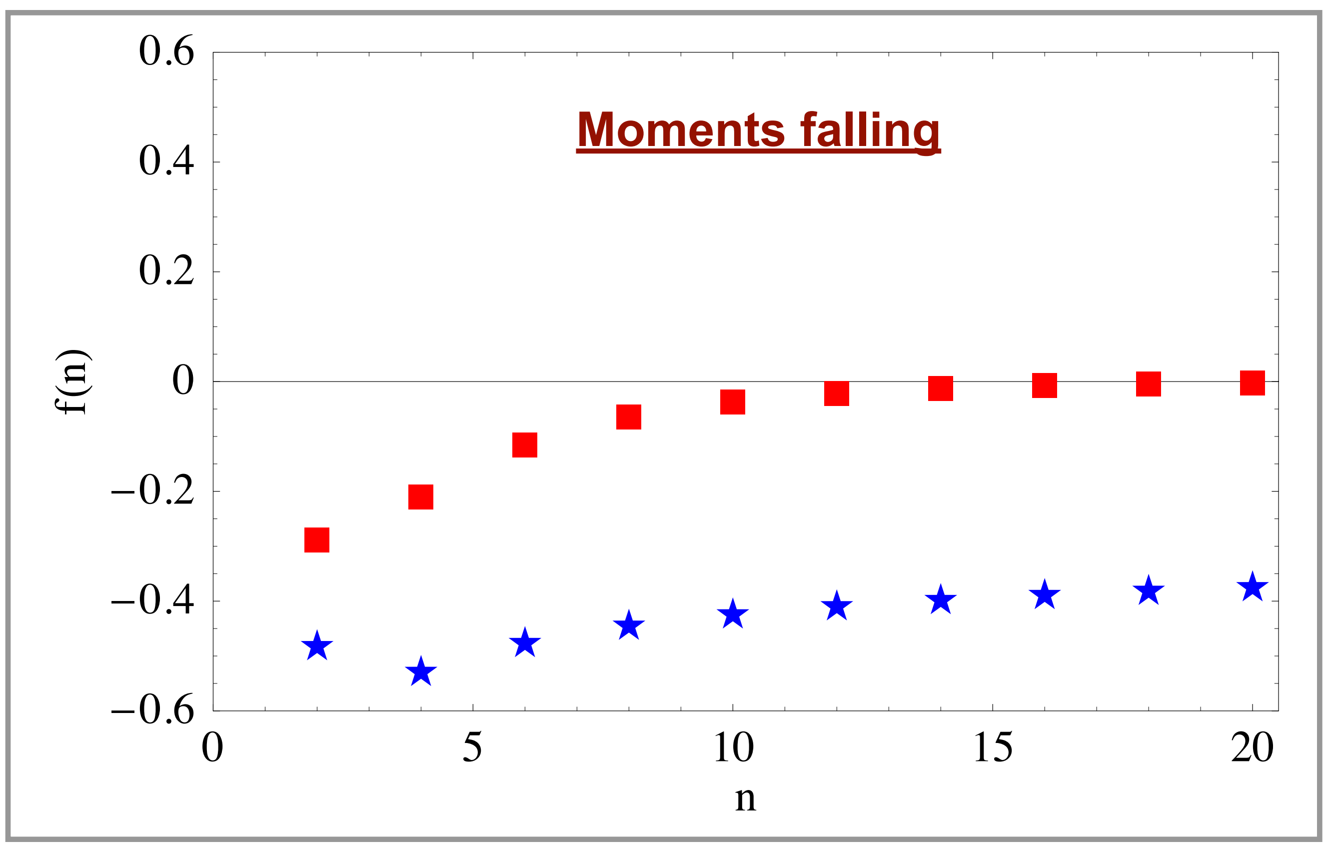
- Scale hierarchy: $\Lambda_{\mathrm{QCD}} \ll |Q|,\, m_\Psi \ll a^{-1}$
  - Naively need fine lattice spacings: $a^{-1} \sim 5\,\mathrm{GeV}$
- Heavy quark quenched: very cheap
  - Use variety of masses
- Different q^μ also easy
- Correlator analysis is straightforward
- Lattice renormalisation and matching is simple

Moment extraction

- Want to extract 6-8 moments
- Two scenarios for n dependence



$M_\Psi = 3.5\,\mathrm{GeV},\ Q^2 = 1.5\,\mathrm{GeV}^2,\ q_0 = 2.8\,\mathrm{GeV}$



$M_\Psi = 2.1\,\mathrm{GeV},\ Q^2 = -3.9\,\mathrm{GeV}^2,\ q_0 = 2.0\,\mathrm{GeV}$

$\alpha = 0.4,\ 1.2\,\mathrm{GeV}, \quad \beta = 0, \quad M_N = 1.2\,\mathrm{GeV}$

Other observables

- Can consider correlator of vector/axial-vector currents (neutrino scattering): odd moments
- Can use unphysical currents
  - Eg: scalar/vector correlator ⇒ moments of transversity distribution
- Moments of GPDs also accessible
- Moments of meson distribution amplitudes

Pion distribution amplitude

- Distribution amplitudes (light-cone WFs) important for QCDF/SCET in hard processes
- Lattice can determine moments of meson distribution amplitudes: e.g. $\phi_\pi(x)$

$$\langle \xi^n \rangle_\pi = \int_0^1 \xi^n \phi_\pi(\xi) d\xi$$

- Local ME method limited to one moment!
- OPE method can determine higher moments
- Not related to a physical process

[see also Braun & Mueller Eur.Phys.J. C55 (2008) 349]

Pion distribution amplitude

- Lattice calculations of the tensor

H-L vector current

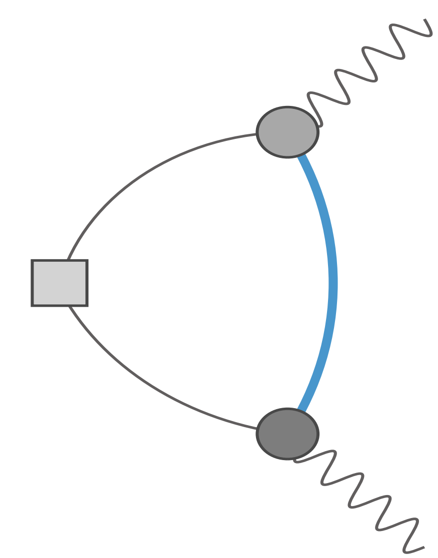
$$S^{\mu\nu}_{\Psi,\psi}(p,q) = \int d^4x \, e^{i\,q\cdot x} \langle \pi^+(p) | T[V^{\mu}_{\Psi,\psi}(x) A^{\nu}_{\Psi,\psi}(0)] | 0 \rangle$$

H-L axial current

- After OPE determine same matrix elements as for distribution amplitude:

$$\langle \pi^+(p) | \overline{\psi} \gamma^{\{\mu_1} \gamma_5 (i\,D)^{\mu_2} \ldots (i\,D)^{\mu_n\}} \psi | 0 \rangle \; = f_\pi \langle \xi^{n-1} \rangle_\pi \, [p^{\mu_1} \ldots p^{\mu_n} - \text{traces}]$$

- Numerical work under consideration
    - Computationally similar to pion FF



Summary

- Information about high x region is difficult to obtain from LQCD but not impossible
- Some possible avenues for progress
- All are significantly more computationally expensive than the current methods for low moments (☹)

[FIN]

Higher twists in OPE

- Derivative squared generates towers of higher twist operators

$$\sum_{n=0}^{\infty} \overline{\psi} \left(\frac{-2i\ q \cdot \overleftrightarrow{D} + \overleftrightarrow{D}^2}{Q^2 - m_\Psi^2} \right)^n \psi \rightarrow \ldots q_{\mu_1} \ldots q_{\mu_3} \overline{\psi} \overleftrightarrow{D}^{\mu_1} \overleftrightarrow{D}^4 \overleftrightarrow{D}^{\mu_2} \overleftrightarrow{D}^{\mu_3} \overleftrightarrow{D}^2 \psi \ldots$$

- Corrections should be $\mathcal{O}\left[\left(\Lambda_{\mathrm{QCD}}^2/Q^2\right)^j\right]$
  - Depend on n since non-Abelian
  - Can be include in lattice analysis
  - Possible to use a free fermion field: no HT

Target mass corrections

- Result from deviation of bound-state from light cone
- OPE basis constructed from operators of definite spin ⇒ trace subtractions
- Eg: $p^{\mu_1} p^{\mu_2} - \mathrm{tr} = p^{\mu_1} p^{\mu_2} - p^2 g^{\mu_1 \mu_2}$

$$p^{\mu_1} \ldots p^{\mu_n} - \mathrm{tr} = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^j (n-j)!}{2^j n!} M^{2j} g^{\{\mu_1 \mu_2} \ldots g^{\mu_{2j-1} \mu_{2j}} p^{\mu_{2j+1}} \ldots p^{\mu_n\}}$$

- Contraction with q^{μ}'s leads to Gegenbauers

Compton correlator

$$G^{\mu\nu}_{(4)}(\mathbf{p},\mathbf{q},t,\tau;\Gamma) = \sum_{\mathbf{x},\mathbf{z}}\sum_{\mathbf{y}} \mathrm{e}^{i\mathbf{p}\cdot\mathbf{x}}\mathrm{e}^{i\mathbf{q}\cdot\mathbf{y}}\Gamma_{\beta\alpha}\langle 0|\chi_\alpha(\mathbf{x},t))\overline{J}^{\mu}_{\Psi,\psi}(\mathbf{y}+\mathbf{z},\tau+\frac{\tau}{2})\overline{J}^{\nu}_{\Psi,\psi}(\mathbf{z},\frac{\tau}{2})\overline{\chi}_\beta(\mathbf{0},0)|0\rangle$$

$$= \sum_{\mathbf{x},\mathbf{z}}\sum_{\mathbf{y}}\sum_{N,N'}\sum_{s,s'} \mathrm{e}^{i(\mathbf{p}-\mathbf{p_N})\cdot\mathbf{x}}\mathrm{e}^{i(\mathbf{p_N}-\mathbf{p_{N'}})\cdot\mathbf{z}}\mathrm{e}^{i\mathbf{q}\cdot\mathbf{y}}\mathrm{e}^{-(E_N+E_{N'})\frac{\tau}{2}}\Gamma_{\beta\alpha}$$

$$\times\langle 0|\chi_\alpha(0)|E_N,\mathbf{p_N},s\rangle\langle E_N,\mathbf{p_N},s|\overline{J}^{\mu}_{\Psi,\psi}(\mathbf{y},\tau)\overline{J}^{\nu}_{\Psi,\psi}(0)|E_{N'},\mathbf{p_{N'}},s'\rangle\langle E_{N'},\mathbf{p_{N'}},s'|\overline{\chi}_\beta(0)|0\rangle$$

$$\xrightarrow{t\to\infty} \mathrm{e}^{-E_0 t}\sum_{\mathbf{y}}\mathrm{e}^{i\mathbf{q}\cdot\mathbf{y}}\sum_{s,s'}\Gamma_{\beta\alpha}\langle 0|\chi_\alpha(0)|E_0,\mathbf{p},s\rangle\langle E_0,\mathbf{p},s|\overline{J}^{\mu}_{\Psi,\psi}(\mathbf{y},\tau)\overline{J}^{\nu}_{\Psi,\psi}(0)|E_0,\mathbf{p},s'\rangle\langle E_0,\mathbf{p},s'|\overline{\chi}_\beta(0)|0\rangle$$

$$G_{(2)}(\mathbf{p},t;\Gamma) = \sum_{\mathbf{x}}\mathrm{e}^{i\mathbf{p}\cdot\mathbf{x}}\Gamma_{\beta\alpha}\langle 0|\chi_\alpha(0)\overline{\chi}_\beta(\mathbf{x},t)|0\rangle$$

$$T^{\{\mu\nu\}}_{\Psi,\psi}(p,q) = 4M\,a\sum_{\tau}\mathrm{e}^{iq_4\tau}\left[\lim_{t\to\infty}\frac{G^{\{\mu\nu\}}_{(4)}(\mathbf{p},\mathbf{q},t,\tau;\Gamma_4)}{G_{(2)}(\mathbf{p},t;\Gamma_4)}\right]$$

F_4 lattice

The F_d lattice in $d \geq 2$ dimensions is defined as the set of points $\{x | x = \sum_\mu x_\mu e_{(\mu)}, x_\mu \in \mathbb{Z}, \sum_\mu x_\mu = \text{even}\}$. Here $e_{(\mu)}$ is the euclidean unit vector in the μ-direction. F_d can be viewed as a hypercubic lattice Z^d with its odd sites removed and for $d = 3$ it corresponds to an *fcc* lattice. The euclidian distance between nearest neighbors is $\sqrt{2}$ and there are $2d(d-1)$ nearest neighbors per site. The discrete rotation symmetry group is particularly large in four dimensions and in this case the Lorentz invariance breaking term in the kinetic energy part of the free propagator vanishes. Invariance breaking terms first occur at order Λ^{-4}.